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BENDING OF WEDGELIKE PLATES WITH ELASTICALLY-FASTENED OR REINFORCED EDGES *

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An exact solution is obtained for a number of problems associated with the investigation of the bending of wedgelike plates with either elastically supported or clamped edges or reinforced by an elastic bar. The following problems are examined: 1) both edges of the plate resist deflections elastically but do not resist rotations; 2) one edge of the plate is rigidly clamped, while the second is elastically resistive to deflection but not resistive to rotation; 3) both edges of a supported plate resist rotation elastically; 4) one edge of the plate is free, while the other is supported and resists rotation elastically; 5) two wedgelike plates with different apex angles and different elastic properties are interconnected by means of an elastic bar operating only in bending. The exact solutions of the problems listed are used to investigate the nature of the singularities in the forces at the angular point of the plate and at infinity.

A method of solving problems on the contact between a semi-infinite beam and an elastic wedge is proposed in /1,2/, which is based on using the Carleman boundary value problem for a strip. The method of /1,2/ is applied to problems 1)— 5) below. Each of the listed problems can be made complicated by assigning inhomogeneous boundary conditions. In this case, the auxiliary problem with classical boundary conditions reduces to a problem on the solution of a homogeneous equation with inhomogeneous nonclassical boundary conditions. Such a transformation is equivalent to replacing the external load by a load acting only on elastically framed edges, and is considered in detail in the example of problem 1). Problems 1)— 5) are examined in Sects. 1—5, respectively.

1. Problem 1) is formulated as follows:

$$\Delta^2 w\ (r,\theta) = q\ (r,\theta)/D, \quad -\alpha \leqslant \theta \leqslant \alpha, \quad 0 \leqslant r < \infty \tag{1.1}$$

$$\theta = \pm\alpha, \quad M_\theta = m_\pm, \quad w - f_\pm = k\ (v_\pm \mp V_\theta)$$

$$\int_0^\infty \Big\{v_+(r) + \sigma_i v_-(r) + k^{-1}[w(r,\alpha) + w(r,-\alpha)]\,\eta_i(\alpha) + \tag{1.2}$$

$$\int_{-\alpha}^{\alpha} q\ (r,\theta)\,\eta_i(\theta)\,r\,d\theta\Big\}\,dr = 0, \quad i = 0, 1, 2$$

$$\sigma_0 = \sigma_1 = -\sigma_2 = 1, \quad \eta_0(\theta) = 1, \quad \eta_1(\theta) = \cos\theta, \quad \eta_2(\theta) = \sin\theta$$

$$M_\theta = -D\left(\frac{\partial}{r\,\partial r} + \frac{\partial^2}{r^2\,\partial\theta^2} + \nu\,\frac{\partial^2}{\partial r^2}\right) w \tag{1.3}$$

$$M_r = -D\left[\frac{\partial^2}{\partial r^2} + \nu\left(\frac{\partial}{r\,\partial r} + \frac{\partial^2}{r^2\,\partial\theta^2}\right)\right] w$$

$$V_\theta = -D\left[\frac{\partial}{r\,\partial\theta}\Delta + (1-\nu)\frac{\partial^2}{\partial r^2}\left(\frac{\partial}{r\,\partial\theta}\right)\right] w$$

$$V_r = -D\left[\frac{\partial}{\partial r}\Delta + (1-\nu)\frac{\partial}{r\,\partial r}\left(\frac{\partial^2}{r\,\partial\theta^2}\right)\right] w$$

Here $w\,(r,\theta)$, ν, D are, respectively, the deflection, Poisson's ratio, and stiffness of the plate, k is the stiffness coefficient of the elastic restraint M_θ, M_r, V_θ, V_r are bending moments and generalised transverse forces, $q\,(r,\theta)$ is a given load acting on the plate, $m_\pm(r)$, $v_\pm(r)$ and $f_\pm(r)$ are respectively, the moments, forces, and initial deflections given on the edges $\theta = \pm\alpha$.

The equilibrium conditions (1.2) assure uniqueness of the solution of the problem posed, which is sought in the form

$$w\ (r,\theta) = w_0\ (r,\theta) + w_1\ (r,\theta) + w_2\ (r,\theta)$$

We have the following equations, boundary conditions, and equilibrium conditions for w_i

$$\Delta^2 w_0\ (r,\theta) = q\ (r,\theta)/D, \quad \Delta^2 w_i\ (r,\theta) = 0 \tag{1.4}$$

$$\theta = \pm\alpha, \quad M_\theta^{(0)} = m_\pm(r), \quad M_\theta^{(i)} = 0$$

$$w_0 = f_\pm + kv_\pm, \quad w_i = \mp k\,(V_\theta^{(1)} - V_*)$$

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$$\int_0^\infty \left[V_+(r) - \frac{1}{k} w_1(r, \alpha)\right] dr = 0$$

$$2\eta_i(\alpha)\int_0^\infty \left[V_* - \frac{1}{k} w_i(r, \alpha)\right] r\, dr = 0, \quad i = 1, 2$$

$$i = 1, \quad V_* = V_+; \quad i = 2, \; V_* = V_-;$$

$$V_\pm(r) = {}^1/_2\,[V_\theta^{(0)}(r, \alpha) \mp V_\theta^{(0)}(r, -\alpha)]$$

The functions $M_\theta^{(i)}$ and $V_\theta^{(i)}$ are determined in terms of w_i by means of (1.3).

To find the function w_0 it is sufficient to apply the Mellin transform (see /3/, for instance)

$$w(p, \theta) = \int_0^\infty w(r, \theta) r^{p-2} dr, \quad w(r, \theta) = \frac{1}{2\pi i}\int_\Omega \bar{w}(p, \theta) r^{1-p} dp \tag{1.5}$$

The scheme in /1,2/ should be used in addition to the transform (1.5) in determining the functions w_i $(i = 1, 2)$.

Let us consider the problem for the function $w_1(r, \theta)$, whose solution we seek in the class of functions possessing an asymptotic $w_1 = o(1)$ for $r \to 0$, and $w_1 = o(r^{-\varepsilon})$, $\varepsilon > 0$ for $r \to \infty$. Taking account of the evenness of this problem and the first of the boundary conditions for w_1, we obtain

$$w_1(r, \theta) = \frac{1-\nu}{8\pi i}\int_\Omega \left[(p+\varkappa)\frac{\cos(p-1)\theta}{\cos(p-1)\alpha} - \right. \tag{1.6}$$

$$\left.(p-1)\frac{\cos(p+1)\theta}{\cos(p+1)\alpha}\right]\Phi(p) r^{1-p} dp$$

Here $\varkappa = (3+\nu)(1-\nu)^{-1}$, $\Omega = \Omega_0$, Ω_n is the line $\operatorname{Re} p = c + 3n$ in the plane of the complex variable p, where the constant c is determined by the class of desired functions, and should be selected from the band $1 < c < 1 + \varepsilon$, in this case.

Assuming $\Phi(p)$ is analytic in the strip Π_0 $(\Pi_n = \{c + 3n < \operatorname{Re} p < c + 3 + 3n\})$, continuous in a closed strip Π_0, and uniformly relative $c \leqslant \sigma \leqslant c + 3$

$$\int_{-\infty}^{\infty} |\Phi(\sigma + it)|^2 dt < \text{const}$$

and requiring that the function (1.6) satisfy the second boundary condition for w_1 from (1.4), we arrive at the Carleman boundary value problem for a strip

$$\Phi(p_0 + 3) - \lambda p_0(p_0^2 - 1)K(p_0)\Phi(p_0) = G(p_0), \quad \operatorname{Re} p_0 = c \tag{1.7}$$

$$\lambda = \frac{kD(1-\nu)}{4(3+\nu)}, \quad K_0(p) = \frac{\sin 2p\alpha - p\varkappa^{-1}\sin 2\alpha}{\cos 2p\alpha + \cos 2\alpha}$$

$$G_\pm(p) = \int_0^\infty V_\pm(r) r^{1+p} dr$$

The operations performed in obtaining the problem (1.7) are legitimate for $V_+(r) r^{c+1/2} \in L_2(0, \infty)$ and $G_0(p) \in H_\Omega$ (H_Ω is the class of functions satisfying the Hölder condition on the line Ω).

By using the function

$$\Psi(p) = \Phi(p)[\lambda^{p/3}\,\Gamma(p-1)\sin \pi p/2]^{-1} \tag{1.8}$$

we write the boundary condition (1.7) in the form

$$\Psi(p_0 + 3) + K(p_0)\Psi(p_0) = G(p_0), \quad \operatorname{Re} p_0 = c \tag{1.9}$$

$$K(p) = -K_0(p)\,\mathrm{tg}\,\pi p/2$$

$$G(p) = -G_+(p)\,[\lambda^{1+p/3}\Gamma(p+2)\cos \pi p/2]^{-1}$$

The desired function $\Psi(p)$ has two simple poles $p_1 = 2$ and $p_2 = 4$ in the strip Π_0. The coefficient $K(p)$ has no zeroes, possesses the asymptotic $K(p) = 1 + o(e^{-2\beta|p|})$, $|p| \to \infty$, $\beta = \min\{\alpha, \pi/2\}$, satisfies the Hölder condition and, moreover, $[\arg K(p)] = 0$. Then according to /1,2/, the solution of the problem (1.9) is given by the formulas

$$\Psi(p) = \mathrm{X}(p)\left[\frac{1}{6i}\int_\Omega \frac{G(s)\,ds}{\mathrm{X}(s+3)\sin\pi(p-s)/3} + \frac{C_1}{\sin\pi(p-2)/3} + \right. \tag{1.10}$$

$$\left.\frac{C_2}{\sin\pi(p-4)/3}\right]$$

$$\mathrm{X}(p) = \exp\left\{\frac{1}{3}\int_\Omega \frac{\ln K(s)\,ds}{\exp[{}^2/_3\,\pi i(s-p)] - 1}\right\}$$

Here C_1 and C_2 are arbitrary constants.

For any integer n the function $\Psi(p)$ is analytic in each strip Π_n with the exception of the points $p_1 = 3n+2$ and $p_2 = 3n+4$, where simple poles are, and it has a jump on each line Ω_n where the limit values to the left of this line $(\Psi_-(p))$ and to the right $(\Psi_+(p) \equiv \Psi(p_0))$ are connected by the relationship

$$\Psi_-(p) = K(p-3n)\Psi_+(p) - (-1)^n G(p-3n), \quad p \in \Omega_n \tag{1.11}$$

$$\Psi_+(p) = X_+(p)\Big[\frac{1}{2}\frac{G(p)}{X(p+3)} + \frac{1}{6i}\int_\Omega \frac{G(s)\,ds}{X(s+3)\sin(p-s)\pi/3} +$$

$$\frac{C_1}{\sin\pi(p-2)/3} + \frac{C_2}{\sin\pi(p-4)/3}\Big]$$

$$X_+(p) = \exp\Big\{-\frac{1}{2}\ln K(p) + \frac{1}{3}\int_\Omega \frac{\ln K(s)\,ds}{\exp[{}^2/_3\pi i(p-s)]-1}\Big\}$$

We determine the arbitrary constants C_1 and C_2 by satisfying the equilibrium conditions (1.4) for w_1, which result, when (1.6), (1.8) and (1.10) are taken into account, in the following equations for these constants:

$$2G_+(-1) + 3k^{-1}\lambda^{2/3}X(2)C_1 = 0; \quad 2\cos\alpha[G_+(0) + \lambda k^{-1}\Psi(3)] = 0 \tag{1.12}$$

The exact solution of the problem (1.4) constructed for w_1 permits determination of the asymptotic of the elastic quantities in the plate. By using the scheme of /1,2/, and (1.3), (1.6), (1.8) and (1.10), we find that the asymptotic

$$w = O(r^{1-\gamma}), \quad M_r = M_\theta = O(r^{-1-\gamma}), \quad V_r = V_\theta = O(r^{-2-\gamma}) \tag{1.13}$$

$$\gamma = -1 + \pi(2\alpha)^{-1}$$

is valid as $r \to \infty$.

Let us investigate the behavior of these quantities at the point of the wedge whose asymptotic as $r \to 0$ is determined by the poles of the integrands of the Mellin integrals obtained in the half-plane $\operatorname{Re} p < c$. Since the expression (1.10) defines the function $\Psi(p)$ which is analytic in the strip Π_0, we apply (1.11) to the solution constrcuted, i.e., we consider $\Psi(p)$ in the strip Π_{-1}. For example, the integral for the quantity M_r transformed in this manner has the form

$$M_r = \frac{D(1-\nu)^2}{8\pi i}\int_\Omega F(p)\Lambda_1(p,\theta)\frac{\cos 2p\alpha + \cos 2\alpha}{\sin 2p\alpha - p\varkappa^{-1}\sin 2\alpha}dp \tag{1.14}$$

$$\Lambda_1(p,\theta) = \Big(p + \frac{3+\nu}{1-\nu}\Big)\frac{\cos(p-1)\theta}{\cos(p-1)\alpha} - \Big(p - \frac{1+3\nu}{1-\nu}\Big)\frac{\cos(p+1)\theta}{\cos(p+1)\alpha}$$

$$F(p) = [\lambda^{p/3}\Gamma(p+2)\Psi_-(p)\cos\pi p/2 - k\lambda^{-1}G_+(p)](p+1)^{-1}r^{-1-p}$$

Investigating the location of the poles of the integrand in (1.14), applying the theorem of residues, and (1.12), we obtain

$$r \to 0, \quad M_r = O(r^{-1+\mu}) \tag{1.15}$$

Here μ is the real part of the root of the equation $\varkappa\sin 2p\alpha - p\sin 2\alpha = 0$, $\operatorname{Re} p > 0$ which is closest to the line $\operatorname{Re} p = 0$. Only such roots are examined in all the transcendental equations to be encountered below. The position of these roots as a function of the angle α is described in /3/.

The asymptotic of the deflections and the transverse forces is found analogously:

$$r \to 0, \quad w = O(1), \quad M_\theta = O(r^{-1+\mu}), \quad V_\theta = V_r = O(r^{-2+\mu}) \tag{1.16}$$

The expressions (1.11) permit determination of the next terms in the expansions of the asymptotics (1.13), (1.15) and (1.16). It follows from (1.16) that as $r \to 0$ the transverse force is bounded only for $\alpha \leqslant \alpha^*$ $(\alpha^* = {}^1/_2\arccos\varkappa^{-1})$. Together with the asymptotics (1.13) and (1.15), this latter permits making the conclusion that the solution of the problem (1.4) for w_1 can be constructed according to the scheme mentioned only for $\alpha \leqslant \alpha^*$ since only in this case is the equilibrium condition for an angular element of the plate satisfied, and also the existence of the integrals (1.5) is assured.

Let us turn to the problem (1.4) for the function w_2 whose solution we seek in the class of functions possessing the asymptotics $w_2 = o(r^\delta)$ as $r \to 0$; $\delta > 0$ and $w_2 = o(r^{-\varepsilon})$, $\varepsilon > -\delta$ as $r \to \infty$. In conformity with this, Ω is the line $\operatorname{Re} p = c$ $(1-\delta < c < 1+\varepsilon)$ in the Mellin integrals (1.5). Analogously to the solution of the problem for w_1, we arrive at the Carleman problem (1.7) with right side $G_-(p)$ and a coefficient having the form

$$K_0(p) = (\sin 2p\alpha + p\varkappa^{-1}\sin 2\alpha)(\cos 2p\alpha - \cos 2\alpha)^{-1} \tag{1.17}$$

Partial factorization of the problem (1.7), (1.17) is realized by the function

$$\Psi(p) = \Phi(p)[\lambda^{p/3}\Gamma(p-1)\cos\pi p/2]^{-1}$$

which has the single pole $p = 3$ in the strip Π_0. Therefore, this function is determined by the expression

$$\Psi(p) = X(p)\left[\frac{1}{6i}\int\limits_{\Omega}\frac{G(s)\,ds}{X(s+3)\sin\pi(p-s)/3} + \frac{C}{\sin\pi p/3}\right] \tag{1.18}$$

The function $X(p)$ is defined in (1.10). The constant C is determined from the equilibrium condition (1.4) for w_2 and is evaluated by the formula

$$C = -2kG(0)\,[3\lambda X(3)]^{-1}$$

Finally, the solution of the problem (1.4) for w_2 is expressed in terms of the function (1.18) as follows

$$w_2(r,\theta) = \frac{1-\nu}{8\pi i}\int\limits_{\Omega}\Lambda_2(p,\theta)\,\lambda^{p/3}\Gamma(p-1)\,\Psi(p)\cos\pi p/2r^{1-p}\,dp \tag{1.19}$$

$$\Lambda_2(p,\ \theta) = (p+\varkappa)\ \ [\sin(p-1)\,\theta/\sin(p-1)\,\alpha - \sin(p+1)\,\theta/\sin(p+1)\,\alpha]$$

The solution (1.19) results in asymptotic behavior of elastic quantities of the form

$$\begin{gathered} r\to\infty,\quad w_2 = O(r^{1-\gamma}),\quad M_r = M_\theta = O(r^{-1-\gamma}),\quad V_r = V_\theta = O(r^{-2-\gamma}) \\ r\to 0,\quad w_2 = O(r^{1+\mu}),\quad M_r = M_\theta = O(r^{-1+\mu}),\quad V_r = V_\theta = O(r^{-2+\mu}) \\ \gamma = -1 + \pi/2 \end{gathered} \tag{1.20}$$

Here μ is the real part of the root of the equation $\varkappa\sin 2p\alpha + p\sin 2\alpha = 0$. As follows from (1.19) and (1.20), in contrast to the problem for w_1 the solution of problem (1.4) for w_2 can be obtained by the scheme described for apex angles $\alpha \leqslant \pi/4 + \alpha^*$. This same scheme is applicable for the solution of problem 1) in the case when lumped forces and moments are applied to the wedge apex.

2. Problem 2) is formulated thus in the simplest case

$$\begin{gathered} \Delta^2 w(r,\theta) = 0 \\ \theta = \alpha,\ w = 0,\ r^{-1}\partial\omega/\partial\theta = 0,\ \theta = 0,\ M_\theta = 0,\ w - k\,V_\theta = v(r) \end{gathered} \tag{2.1}$$

The solution of problem (2.1) is sought in the class of functions possessing the asymptotic $w = o(r^{\delta})$, $\delta > -1$ as $r\to 0$, and $w = o(r^{-\varepsilon})$, $\varepsilon > -\delta$ as $r\to\infty$, in the form of the Mellin integral

$$w(r,\theta) = \frac{1}{2\pi i}\int\limits_{\Omega}[A_1\cos(p-1)\theta + A_2\cos(p+1)\theta + B_1\sin(p-1)\theta + B_2\sin(p+1)\theta]\,r^{1-p}\,dp \tag{2.2}$$

Here A_j, B_j $(j = 1,2)$ are determined from the boundary conditions by the following relationships

$$A_1 = -A_2(p+\varkappa)\ (p-\varkappa)^{-1},\ B_j = b_j b^{-1} A_2,\ j = 1,2$$

$$A_2 = 4\,(1-\nu)^{-1}\lambda^{p/3}\Gamma(p)\ \Psi_+(p)\ \sin\pi p/2$$

$$\Psi(p) = X(p)\left[\frac{1}{6i}\int\limits_{\Omega}\frac{G(s)\,ds}{X(s+3)\sin\pi(p-s)/3} - \frac{C}{\sin(p-2)\pi/2}\right]$$

$$b_1 = -p - 1 + (p-1)^{-1}(p+\varkappa)(\cos 2p\alpha + p\cos 2\alpha)$$

$$b_2 = -\cos 2\alpha p + p\cos 2\alpha - p - \varkappa,\ b = \sin 2\alpha p - p\sin 2\alpha$$

The function $X(p)$ is defined in (1.10), where the functions $K(p)$ and $G(p)$ are determined by (1.9) in which we should set

$$K_0(p) = -b^{-1}\,[\cos 2\alpha p - 2\varkappa^{-1}p^2\sin^2\alpha + (\varkappa^2+1)\,(2\varkappa)^{-1}]$$

$$G_+(p) = \int\limits_0^{\infty} v(r)\,r^{1+p}\,dp$$

The arbitrary constant C is determined by the condition for correctness of the operations $\Psi(1) = 0$ performed during solution of the problem.

The exact solution constructed results in the asymptotic formulas (1.20) in which γ and μ are the real parts of roots of the appropriate equations $(\sin 2p\alpha - p\sin 2\alpha)\,(p-1)^{-1} = 0$ and $\varkappa\sin^2 p\alpha + p^2\sin^2\alpha - (1+\varkappa)^2/4 = 0$. It is seen from the asymptotic found that the solution of the problem (2.1) can be sought in the form of the Mellin integral (2.2) for any α. Let us note that despite the unboundedness of the bending moment and the generalized transverse force at the plate apex (for those α for which $\mu < 2$), the equilibrium conditions are satisfied for each element containing an angular point, i.e., the forces originating in the plate are self-equilibrated. In addition to the problem (2.1), a plate bending can be formulated where one edge is elastically supported $(M_\theta = 0,\ w - kV_\theta = v(r))$, while one of the classic conditions $r^{-1}\partial w/\partial\theta = V_\theta = 0$; $M_\theta = w = 0$; $M_\theta = V_\theta = 0$ is given on the other. However, there is no need to consider the first two problems, specially since they can be treated as the problem (1.4) for w_1 and w_2, respectively, for half a plate.

3. In the simplest case the axisymmetric component of the problem is equivalent to the

construction of a biharmonic function satisfying the boundary conditions

$$\theta = \pm\alpha,\ w = 0,\ M_\theta \pm kr^{-1}\,\partial w/\partial\theta = m_\pm(r) \qquad (3.1)$$

in the domain $0 \leqslant r < \infty,\ -\alpha \leqslant \theta \leqslant \alpha$. Here k is the stiffness coefficient of the elastic support, and $m_\pm(r)$ is the moment loading applied to the boundary.

The solution of the problem in the class of functions possessing the asymptotic

$$r \to 0,\quad w = o(r^\delta),\quad \delta > 0 \text{ and } r \to \infty,\quad w = o(r^{-\varepsilon}),\quad \varepsilon > -\delta$$

has the form

$$w(r,\theta) = \frac{1}{2\pi i}\int_\Omega L_+(p,\theta)\,\Gamma(p)\,\lambda^p K^+(p)\,\Psi_+(p)\,r^{1-p}\sin\pi p/2\,dp \qquad (3.2)$$

$$L_+(p,\theta) = \cos(p-1)\theta/\cos(p-1)\alpha - \cos(p+1)\theta/\cos(p+1)\alpha$$

$$K^\pm(p) = (\cos 2p\alpha \pm \cos 2\alpha)(\sin 2p\alpha \pm p\sin 2\alpha)^{-1},\ \lambda = 4Dk^{-1}$$

Here the function, analytic in the strip $c \leqslant \operatorname{Re} p \leqslant c + 1$, is a solution of the following Carleman problem:

$$\Psi(p+1) + K(p)\,\Psi(p) = G(p),\ K(p) = K^+(p)\operatorname{tg}\pi p/2 \qquad (3.3)$$

$$G(p) = G_+(p)\,[\lambda^{p+1}\Gamma(p+1)\cos\pi p/2]^{-1},\quad G_\pm(p) = \int_0^\infty m_\pm(r)\,r^{p-1}dr$$

$$\Psi(p) = X(p)\frac{1}{2i}\int_\Omega \frac{G(s)\,ds}{X(s+1)\sin\pi(p-s)} \qquad (3.4)$$

$$X(p) = \exp\left\{\int_\Omega \frac{\ln K(s)\,ds}{\exp[2\pi i(s-p)]-1}\right\}$$

The asymptotic expressions for the quantities are determined by (1.20) in which γ is the real part of the root of the equation $\sin 2p\alpha + p\sin 2\alpha = 0$ and $\mu = -1 + \pi(2\alpha)^{-1}$.

This asymptotic shows that the solution of the problem (3.1) can be sought in the form (3.2) for $\alpha < \pi/4$.

The formulation of the problem 3) for the antisymmetric component differs from the symmetric case only by the boundary conditions

$$\theta = \pm\alpha,\ w = 0,\ \pm M_\theta - kr^{-1}\,\partial w/\partial\theta = m_-(r)$$

The following integral yields its solution

$$w(r,\theta) = \frac{1}{2\pi i}\int_\Omega L_-(p,\theta)\,\Gamma(p)\,\lambda^p K^-(p)\,\Psi_+(p)\,r^{1-p}\sin\pi p/2\,dp \qquad (3.5)$$

$$L_-(p,\theta) = \sin(p-1)\theta/\sin(p-1)\alpha - \sin(p+1)\theta/\sin(p+1)\alpha$$

Here the function $\Psi(p)$ is determined by (3.3) and (3.4), and the function $K^-(p)$ is determined in (3.2).

The asymptotic of the problem has the form (1.20) in which γ is the appropriate solution of the equation $(p-1)^{-1}(\sin 2p\alpha - p\sin 2\alpha) = 0$ and $\mu = -1 + \pi\alpha^{-1}$.

We note that (3.5) yields the solution of the antisymmetric component of problem 3) for $\alpha < \pi/2$.

4. Let us consider the bending problem for a plate with one edge supported and elastically resistive to rotation by giving one of the classic boundary conditions

$$r^{-1}\,\partial w/\partial\theta = V_\theta = 0,\ w = M_\theta = 0,\ w = r^{-1}\,\partial w/\partial\theta = 0$$
$$M_\theta = V_\theta = 0$$

on the second edge.

The first two variants can be considered as symmetric and antisymmetric components of the problem 3) for half a plate. Let us examine just the last variant. In the simplest case it is equivalent to constructing a biharmonic function $w(r,\theta)$ satisfying the conditions

$$\theta = \alpha,\ M_\theta = V_\theta = 0;\ \theta = 0,\ w = 0,\ kr^{-1}\,\partial w/\partial\theta + M_\theta = m(r)$$

$$\int_0^\infty \left[m(r) + kr^{-1}\frac{\partial w}{\partial\theta}(r,0)\right]dr = 0$$

in the domain $0 \leqslant r < \infty,\ 0 \leqslant \theta \leqslant \alpha$. Here $m(r)$ is the moment loading applied to the plate edge $\theta = 0$. We write the solution of this problem in the form of a Mellin integral

$$w(r,\theta) = \frac{1}{2\pi i}\int_\Omega L_*(p,\theta)\,a^{-1}\lambda^p\Gamma(p)\,\Psi_+(p)\,r^{1-p}\cos\pi p/2\,dp$$

$$L_*(p,\theta) = \cos(p+1)\theta - \cos(p-1)\theta + a_1 \sin(p+1)\theta + a_2 \sin(p-1)\theta$$

$$a = \sin^2 p\alpha + p^2 \varkappa^{-1} \sin^2\alpha - (1+\varkappa)^2 (4\varkappa)^{-1}$$

$$a_1 = \cos 2p\alpha + p\varkappa^{-1} \cos 2\alpha - \varkappa^{-1}(p-1) \quad a_2 = \cos 2p\alpha - p\varkappa^{-1} \cos 2\alpha + \varkappa^{-1}(p-1)^{-1}(p^2 - \varkappa^2)$$

$$\Psi(p) = X(p)\left[\frac{1}{2i}\int_\Omega \frac{G(s)\,ds}{X(s+1)\sin\pi(p-s)} + \frac{C}{\sin\pi(p-1)}\right]$$

$$C = \frac{G_0(0)}{2\pi D}$$

$$G(p) = -G_0(p)\,[\lambda^{p+1}\Gamma(1+p)\sin\pi p/2]^{-1}, \quad G_0(p) = \int_0^\infty m(r)\, r^{p-1}\, dr$$

The contour of integration Ω is selected exactly as in problem 3), and the function $X(p)$ is determined in (3.4), where

$$K(p) = -(\sin 2p\alpha - p\varkappa^{-1}\sin 2\alpha)\, a^{-1} \operatorname{ctg} \pi p/2$$

The asymptotic of problem 4) is determined by (1.20), where γ and μ are, respectively, the real parts of roots of the equations

$$\varkappa \sin^2 \alpha p + p^2 \sin^2\alpha - (1+\varkappa)^2/4 = 0$$
$$\varkappa \sin 2p\alpha - p \sin 2\alpha = 0$$

5. Let us investigate the bending problem of two wedgelike plates occupying the domain A: $(0 \leqslant r < \infty, -\beta \leqslant \theta \leqslant 0)$ and B: $(0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \alpha)$ which are hinge-supported along the edges $\theta = -\beta$ and $\theta = \alpha$ and connected by a bar not operating under torsion. The minus superscript denotes elastic quantities in the domain A, while the plus superscript denotes quantities in domain B. The problem under consideration reduces to the construction of two biharmonic functions $w^-(r,\theta)$ and $w^+(r,\theta)$ satisfying the following conditions

$$\theta = -\beta,\ w^- = M_\theta^- = 0;\ \theta = \alpha,\ w^+ = M_\theta^+ = 0 \tag{5.1}$$
$$\theta = 0,\ D_0\, \partial^4 w / \partial r^4 = V_\theta^+ - V_\theta^- + g(r),\ w^- = w^+ = w$$
$$\partial\omega^- / \partial\theta = \partial w^+ / \partial\theta,\ M_\theta^- = M_\theta^+$$

in the domains A and B. Here w is the beam deflection, and $q(r)$ is the load acting on it. The solution of the problem (5.1) in the class of functions $r \to 0$, $w^\pm = o(r^\varepsilon)$, $\varepsilon > 1$ and $r \to \infty$, $w^\pm = o(r^\delta)$, $\varepsilon > -\delta$ is respresentable in the form of a Mellin integral

$$w^\pm(r,\theta) = \frac{1}{2\pi i}\int_\Omega R^\pm(p,\theta)\,\lambda^p\Gamma(p-1)\,\Psi_+(p)\, r^{1-p}\cos\pi p/2\, dp \tag{5.2}$$

$$R^\pm(p,\theta) = A_1\cos(p-1)\theta + A_2\cos(p+1)\theta + B_1^\pm \sin(p-1)\theta + B_2^\pm \sin(p+1)\theta$$

$$A_1 = a_1 \sin(p-1)\alpha\sin(p-1)\beta,\ A_2 = -a_2\sin(p+1)\alpha \times \sin(p+1)\beta$$

$$B_1^- = a_1\sin(p-1)\alpha\cos(p-1)\beta,\ B_2^- = -a_2\sin(p+1)\alpha\times\cos(p+1)\beta$$

$$B_1^+ = -a_1\cos(p-1)\alpha\sin(p-1)\beta,\ B_2^+ = a_2\cos(p+1)\alpha\times\sin(p+1)\beta$$

$$a_1 = (p+1)\operatorname{cosec}[(\alpha+\beta)(p-1)],\quad a_2 = (p-1)\operatorname{cosec}[(\alpha+\beta)(p+1)]$$

$$\lambda = \frac{D_0}{4D_1},\quad \Psi(p) = X(p)\left[\frac{1}{2i}\int_\Omega \frac{G(s)\,ds}{X(s+1)\sin\pi(p-s)}\right]$$

$$X(p) = \exp\left\{\int_\Omega \frac{\ln K(s)\,ds}{\exp[2\pi i(p-s)] - 1}\right\},\quad G(s) = \frac{Q(p)}{D_0\lambda^p\Gamma(p+3)\sin\pi p/2}$$

$$Q(p) = \int_0^\infty g(r)\, r^{p+2}\, dr,\quad K(p) = a_1 a_2 (p^2-1)^{-1}(A_1 + A_2)\operatorname{ctg}\pi p/2$$

The asymptotic of the elastic quantities has the form (1.20), where $\gamma = -1 + \pi(\alpha+\beta)^{-1}$ and μ is the real part of the root of the equation

$$\sin 2p(\alpha+\beta) - \cos 2\alpha \sin 2p\beta - \cos 2\beta\sin 2p\alpha + p\,[\sin 2(\alpha+\beta) - \sin 2\alpha\cos 2p\beta - \sin 2\beta \cos 2p\alpha] = 0$$

Besides the problem 5) considered, a cycle of problems can be mentioned about the bending of wedgelike plates reinforced by elastic bars whose exact solution is constructed by the method elucidated. Among this cycle and problems in which the bar having only finite bending stiffness differs by the contact conditions on the edge $\theta = 0$. For $\theta = 0$ the first two conditions in (5.1) are common and the last can be replaced by any of the following pairs of conditions: 1) $r^{-1}\partial w^+ / \partial\theta = r^{-1}\partial w^- / \partial\theta = 0$, 2) $M_\theta^+ = M_\theta^- = 0$, 3) $r^{-1}\partial w^+ / \partial\theta = M_\theta^- = 0$ or $M_\theta^+ = r^{-1}\partial w^- / \partial\theta = 0$.

The mentioned method is also used to construct the solution of problems in which there is a bar having a finite torsional stiffness at the edge $\theta = 0$ instead of the bar having the finite bending stiffness. For such bars the general conditions are

$$r^{-1}\partial w^- / \partial\theta = r^{-1}\partial w^- / \partial\theta; \quad G_0 r^{-1}\partial w / \partial\theta = M_\theta^+ - M_\theta^- + m(r)$$

to which any of the following pairs of conditions must be appended:

$$1)\ w^+ = w^-,\ V_\theta^+ = V_\theta^-; \quad 2)\ w^+ = w^- = 0$$

$$3)\ V_\theta^+ = V_\theta^- = 0; \quad 4)\ w^+ = V_\theta^- = 0$$

or $w^- = V_\theta^+ = 0$. In all these problems, the hinge-support conditions on the edges $\theta = -\beta, \theta = \alpha$ can be replaced by other classical conditions which are not absolutely identical on both edges. Moreover, the materials of the wedges A and B can have different elastic properties up to anisotropy. Among this cycle of problems are also problems on the bending of a wedgelike plate one of whose edges is clamped classically while the other is reinforced by an elastic bar (beam), where this reinforcement is described by the conditions

$$D_0 \partial^4 w / \partial r^4 = g(r) - V_\theta, \quad M_\theta = m(r)$$

This last condition can be replaced by the condition $r^{-1}\partial w / \partial\theta = \varphi(r)$.

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